On the Logical Shadow of the Explicit Constructions

Amir Akbar Tabatabai

Department of Philosophy, Utrecht University

Oberwolfach Workshop
Mathematical Logic: Proof Theory, Constructive Mathematics 2020

Logic as the Shadow of Constructions

"It is equally stupid and simple to consider mathematics to be just an axiom system as it is to see a tree as nothing but a quantity of planks." L.E.J. Brouwer

Logic as the Shadow of Constructions

> *"It is equally stupid and simple to consider mathematics to be just an axiom system as it is to see a tree as nothing but a quantity of planks."* L.E.J. Brouwer

According to Brouwer, the main player in mathematics is *construction* and logic is only the set of its universally *constructible/provable* statements. Taking logic as the foundation is just a classical habit that fixes the logic at first and add some axioms on top of it. What if the logic changes after adding the axioms? If so, which one is the real logic of your system? We do not encounter this problem in the classical world, since classical logic is a maximal consistent logic.

Logic as the Shadow of Constructions

> *"It is equally stupid and simple to consider mathematics to be just an axiom system as it is to see a tree as nothing but a quantity of planks."* L.E.J. Brouwer

According to Brouwer, the main player in mathematics is *construction* and logic is only the set of its universally *constructible/provable* statements. Taking logic as the foundation is just a classical habit that fixes the logic at first and add some axioms on top of it. What if the logic changes after adding the axioms? If so, which one is the real logic of your system? We do not encounter this problem in the classical world, since classical logic is a maximal consistent logic.

Logic is not the foundation of a discourse. It is just its shadow!

Logic as the Shadow of Constructions

> *"It is equally stupid and simple to consider mathematics to be just an axiom system as it is to see a tree as nothing but a quantity of planks."* L.E.J. Brouwer

According to Brouwer, the main player in mathematics is *construction* and logic is only the set of its universally *constructible/provable* statements. Taking logic as the foundation is just a classical habit that fixes the logic at first and add some axioms on top of it. What if the logic changes after adding the axioms? If so, which one is the real logic of your system? We do not encounter this problem in the classical world, since classical logic is a maximal consistent logic.

Logic is not the foundation of a discourse. It is just its shadow!

To formalize Brouwer's interpretation of logic, we need to introduce **constructions** at first and then the **interpretation** of the formulas via this given notion of constructibility.

The Constructions

What is a construction?

The Constructions

What is a construction?

- a computable function in $\mathbb{N}$ or HA,
- a definable function in HA or HA^ω,
- a function in IZF or CZF,
- a term in Martin Löf type theory,
- a morphism in a strong enough category like a locally Cartesian closed category or a topos.

The Constructions

What is a construction?

- a computable function in $\mathbb{N}$ or HA,
- a definable function in HA or HA^ω,
- a function in IZF or CZF,
- a term in Martin Löf type theory,
- a morphism in a strong enough category like a locally Cartesian closed category or a topos.

We use IZF, for simplicity. IZF is a system in the usual language of set theory, i.e., $\mathcal{L} = \{\in\}$, using the intuitionistic logic and the Zermelo-Frankel axioms, except for the foundation axiom which is replaced by the following set-induction:

$$\forall x[\forall y \in x A(y) \to A(x)] \to \forall x A(x)$$

and the replacement axiom replaced by the collection axiom.

A *Heyting interpretation* is a map that assignments two sets to any proposition A, the set of its *possible constructions*, denoted by $[A]_1$ and the set of its *actual constructions*, denoted by $[A]_0$ such that:

A Spectrum of Interpretations: Heyting's Interpretation

A *Heyting interpretation* is a map that assignments two sets to any proposition A, the set of its *possible constructions*, denoted by $[A]_1$ and the set of its *actual constructions*, denoted by $[A]_0$ such that:

- $[p]_1$ and $[\bot]_1$ are inhabited, $[p]_0 \subseteq [p]_1$, for any atomic formula p and $[\bot]_0 = \emptyset$,
- $[A \wedge B]_1 = [A]_1 \times [B]_1$ and
  $[A \wedge B]_0 = \{(x, y) \in [A \wedge B]_1 \mid x \in [A]_0 \wedge y \in [B]_0\}$,
- $[A \vee B]_1 = [A]_1 + [B]_1$ and
  $[A \vee B]_0 = \{(i, x) \in [A \vee B]_1 \mid [i = 0 \to x \in [A]_0] \wedge [i = 1 \to x \in [B]_0]\}$,
- $[A \to B]_1 = [B]_1^{[A]_1}$ and
  $[A \to B]_0 = \{f \in [A \to B]_1 \mid \forall x \in [A]_0 \; f(x) \in [B]_0\}$.

A Spectrum of Interpretations: Brouwer's Interpretation

A *Brouwer's interpretation* is defined exactly in the same way as Heyting's, except for the disjunction:

$[A \vee B]_1 = \|[A]_1 + [B]_1\|$, where $\| - \|$ is the propositional truncation, i.e., $\|X\| = \{x \in \{0\} \mid \exists y \in X\}$ and $[A \vee B]_0 = \{x \in \{0\} \mid \exists y \in [A]_0 \vee \exists y \in [B]_0\}$.

A Spectrum of Interpretations: Brouwer's Interpretation

A *Brouwer's interpretation* is defined exactly in the same way as Heyting's, except for the disjunction:

$[A \vee B]_1 = \|[A]_1 + [B]_1\|$, where $\| - \|$ is the propositional truncation, i.e., $\|X\| = \{x \in \{0\} \mid \exists y \in X\}$ and
$[A \vee B]_0 = \{x \in \{0\} \mid \exists y \in [A]_0 \vee \exists y \in [B]_0\}$.

Given a construction of a disjunction:

A Spectrum of Interpretations: Brouwer's Interpretation

A *Brouwer's interpretation* is defined exactly in the same way as Heyting's, except for the disjunction:

$[A \vee B]_1 = \|[A]_1 + [B]_1\|$, where $\| - \|$ is the propositional truncation, i.e., $\|X\| = \{x \in \{0\} \mid \exists y \in X\}$ and
$[A \vee B]_0 = \{x \in \{0\} \mid \exists y \in [A]_0 \vee \exists y \in [B]_0\}$.

Given a construction of a disjunction:

- Heyting: Total decidability of which disjunct is provable and a direct access to the corresponding proof.

A Spectrum of Interpretations: Brouwer's Interpretation

A *Brouwer's interpretation* is defined exactly in the same way as Heyting's, except for the disjunction:

$[A \vee B]_1 = \| [A]_1 + [B]_1 \|$, where $\| - \|$ is the propositional truncation, i.e., $\|X\| = \{x \in \{0\} \mid \exists y \in X\}$ and
$[A \vee B]_0 = \{x \in \{0\} \mid \exists y \in [A]_0 \vee \exists y \in [B]_0\}$.

Given a construction of a disjunction:

- Heyting: Total decidability of which disjunct is provable and a direct access to the corresponding proof.
- Brouwer: No non-trivial information about the provable disjunct or its corresponding proof.

Three Types of Interpretations

Standing anywhere in the spectrum, it is also possible to restrict yourself to a subclass of the interpretations:

Definition

An interpretation is called:

- **Markov**, if $\neg\neg\exists x \in [p]_0 \to \exists x \in [p]_0$,

Three Types of Interpretations

Standing anywhere in the spectrum, it is also possible to restrict yourself to a subclass of the interpretations:

Definition

An interpretation is called:

- **Markov**, if $\neg\neg\exists x \in [p]_0 \to \exists x \in [p]_0$,
- **Kolmogorov**, if $[p]_1$ is an external finite set and $\neg\neg(x \in [p]_0) \to (x \in [p]_0)$,

Three Types of Interpretations

Standing anywhere in the spectrum, it is also possible to restrict yourself to a subclass of the interpretations:

Definition

An interpretation is called:

- **Markov**, if $\neg\neg\exists x \in [p]_0 \to \exists x \in [p]_0$,
- **Kolmogorov**, if $[p]_1$ is an external finite set and $\neg\neg(x \in [p]_0) \to (x \in [p]_0)$,
- **Proof-irrelevant**, if inhabited-ness of $[p]_0$ implies $[p]_0 = [p]_1$.

Finally, with the appropriate notions of construction and interpretation, we are ready to formalize what we mean by the theory and the logic of a calculus of constructions via the given interpretation:

Finally, with the appropriate notions of construction and interpretation, we are ready to formalize what we mean by the theory and the logic of a calculus of constructions via the given interpretation:

Definition

Let $\mathcal{C}$ be a definable class of Heyting interpretations. By the $\mathcal{C}$-Heyting theory of IZF, denoted by $\mathbf{T}^H_{\mathcal{C}}(\mathsf{IZF})$, we mean the set of all propositional formulas A such that $\mathsf{IZF} \vdash \forall [-] \in \mathcal{C} \; \exists x \in [A]_0$, and by $\mathbf{L}^H_{\mathcal{C}}(\mathsf{IZF})$, we mean the set of all propositional formulas A such that $\sigma(A) \in \mathbf{T}^H_{\mathcal{C}}(\mathsf{IZF})$, for any propositional substitution σ. Similarly, define $\mathcal{C}$-Brouwer theory and logic of IZF, denoted by $\mathbf{T}^B_{\mathcal{C}}(\mathsf{IZF})$ and $\mathbf{L}^B_{\mathcal{C}}(\mathsf{IZF})$, respectively.

The Theory and the Logic of a Mathematical World

Finally, with the appropriate notions of construction and interpretation, we are ready to formalize what we mean by the theory and the logic of a calculus of constructions via the given interpretation:

Definition

Let $\mathcal{C}$ be a definable class of Heyting interpretations. By the $\mathcal{C}$-Heyting theory of IZF, denoted by $\mathbf{T}^H_{\mathcal{C}}(\mathrm{IZF})$, we mean the set of all propositional formulas A such that $\mathrm{IZF} \vdash \forall[-] \in \mathcal{C}\ \exists x \in [A]_0$, and by $\mathbf{L}^H_{\mathcal{C}}(\mathrm{IZF})$, we mean the set of all propositional formulas A such that $\sigma(A) \in \mathbf{T}^H_{\mathcal{C}}(\mathrm{IZF})$, for any propositional substitution σ. Similarly, define $\mathcal{C}$-Brouwer theory and logic of IZF, denoted by $\mathbf{T}^B_{\mathcal{C}}(\mathrm{IZF})$ and $\mathbf{L}^B_{\mathcal{C}}(\mathrm{IZF})$, respectively.

For the definable classes of Markov, Kolmogorov and proof-irrelevant interpretations, we use M, K and PI, respectively. Moreover, whenever we mean a Heyting interpretation, we may drop the superscript H.

The Characterization of Brouwer's Logic

Theorem

$\mathbf{T}^B(\mathsf{IZF}) = \mathsf{IPC}$.

Theorem

$\mathbf{T}^B(\mathrm{IZF}) = \mathrm{IPC}$.

Proof.

The soundness is easy! For the completeness, let τ be a set-theoretical substitution and set $[p]_1 = [\bot]_1 = \{0\}$ and $[p]_0 = \{x \in \{0\} \mid \tau(p)\}$. It is easy to see that any $[A]_1$ has exactly one canonical element. Call it θ_A. Then it is also easy to see that $\theta_A \in [A]_0$ iff $\tau(A)$. Therefore, $\mathrm{IZF} \vdash \tau(A)$, for any set-theoretical substitution. By the recent beautiful theorem by **Robert Passman**, we have $\mathrm{IPC} \vdash A$. □

Other Brouwerian Corollaries

Remark

Note that Brouwer's interpretation is just the *truth-value computation* and hence the Brouwer's logic of a theory is its propositional logic in the usual sense.

Other Brouwerian Corollaries

Remark

Note that Brouwer's interpretation is just the *truth-value computation* and hence the Brouwer's logic of a theory is its propositional logic in the usual sense.

The followings are the consequences of the previous theorem:

Corollary

- *Proof-irrelevant:* $\mathbf{T}^B_{PI}(\mathsf{IZF}) = \mathsf{IPC}$.
- *Markov and Proof-irrelevant (Theory):* $\mathbf{T}^B_{MPI}(\mathsf{IZF}) = \mathsf{IPC}^n = \mathsf{IPC} + \{\neg\neg p \to p \mid p \text{ is an atom}\}$.
- *Markov and Proof-irrelevant (Logic):* $\mathbf{L}^B_{MPI}(\mathsf{IZF}) = \mathsf{IPC}$.
- *Kolmogorov:* $\mathbf{L}^B_{K}(\mathsf{IZF}) = \mathsf{IPC}$.

Heyting Diverges!

Kreisel-Putnam Logic:

$$\mathsf{KP} = \mathsf{IPC} + (\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C).$$

Heyting Diverges!

Kreisel-Putnam Logic:

$$\mathsf{KP} = \mathsf{IPC} + (\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C).$$

Theorem

$\mathbf{T}^H(\mathsf{IZF}) \supseteq \mathsf{KP}$. *Therefore,* $\mathbf{T}^H(\mathsf{IZF}) \neq \mathsf{IPC}$.

Kreisel-Putnam Logic:

$$\mathrm{KP} = \mathrm{IPC} + (\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C).$$

Theorem

$\mathbf{T}^H(\mathrm{IZF}) \supseteq \mathrm{KP}$. *Therefore,* $\mathbf{T}^H(\mathrm{IZF}) \neq \mathrm{IPC}$.

Proof.

The core ideas behind the Heyting validity of the axiom KP are:

Heying Diverges!

Kreisel-Putnam Logic:

$$\mathrm{KP} = \mathrm{IPC} + (\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C).$$

Theorem

$\mathbf{T}^H(\mathrm{IZF}) \supseteq \mathrm{KP}$. *Therefore,* $\mathbf{T}^H(\mathrm{IZF}) \neq \mathrm{IPC}$.

Proof.

The core ideas behind the Heyting validity of the axiom KP are:

- The explicit information encoded in any proof of a disjunction,

Heyting Diverges!

Kreisel-Putnam Logic:

$$\mathsf{KP} = \mathsf{IPC} + (\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C).$$

Theorem

$\mathbf{T}^H(\mathsf{IZF}) \supseteq \mathsf{KP}$. *Therefore,* $\mathbf{T}^H(\mathsf{IZF}) \neq \mathsf{IPC}$.

Proof.

The core ideas behind the Heyting validity of the axiom KP are:

- The explicit information encoded in any proof of a disjunction,
- The fact that $\neg A$ has the following property: If $a \in [\neg A]_1$ and $x \in [\neg A]_0$ then $a \in [\neg A]_0$.

□

Going Beyond KP

The second condition is nothing but proof-irrelevancy as we already expected from the negative formulas. In fact, if we have a proof-irrelevant interpretation and a $\vee$-free formula A, then $a \in [A]_1$ and $x \in [A]_0$ implies $a \in [A]_0$. Therefore, we can use the same construction as we had for the axiom KP to show that the proof-irrelevant interpretations validate the axiom:

$$(A \to B \vee C) \to (A \to B) \vee (A \to C)$$

where A is $\vee$-free.

The Independence of Almost Negative Premises

Definition

Independence of Almost Negative Premises:

$$\mathsf{INP} = \mathsf{IPC} + (A \to B \vee C) \to (A \to B) \vee (A \to C)$$

where A is $\vee$-free.

The Independence of Almost Negative Premises

Definition

Independence of Almost Negative Premises:

$$\mathsf{INP} = \mathsf{IPC} + (A \to B \vee C) \to (A \to B) \vee (A \to C)$$

where A is $\vee$-free.

Note that INP is a theory and not a logic, since it is not closed under substitution. Moreover, note that $\mathbf{L}(\mathsf{INP}) \supseteq \mathsf{KP}$, because in any substitution for $(\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C)$, the formula $\neg A$ is IPC-equivalent to an almost negative formula.

The Independence of Almost Negative Premises

Definition

Independence of Almost Negative Premises:

$$\mathsf{INP} = \mathsf{IPC} + (A \to B \vee C) \to (A \to B) \vee (A \to C)$$

where A is $\vee$-free.

Note that INP is a theory and not a logic, since it is not closed under substitution. Moreover, note that $\mathbf{L}(\mathsf{INP}) \supseteq \mathsf{KP}$, because in any substitution for $(\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C)$, the formula $\neg A$ is IPC-equivalent to an almost negative formula.

Conjecture

$\mathbf{L}(\mathsf{INP}) = \mathsf{KP}$.

The Heyting Characterizations

Theorem

$\mathbf{T}_{PI}(\mathsf{IZF}) = \mathsf{INP}$.

The Heyting Characterizations

Theorem

$\mathbf{T}_{PI}(\mathrm{IZF}) = \mathrm{INP}$.

- **Medvedev Logic:** ML is the logic of the Kripke frames $(P(\{0, \ldots, n\}) - \{\{0, \cdots, n\}\}, \subseteq)$.

The Heyting Characterizations

Theorem

$\mathbf{T}_{PI}(\mathsf{IZF}) = \mathsf{INP}$.

- **Medvedev Logic:** ML is the logic of the Kripke frames $(P(\{0, \ldots, n\}) - \{\{0, \cdots, n\}\}, \subseteq)$.
- ML was originally introduced as the characterization of Kolmogorov's logic of finite problems. Hence, it is expected to be a relevant logic here, as well.

The Heyting Characterizations

Theorem

$\mathbf{T}_{PI}(\text{IZF}) = \text{INP}$.

- **Medvedev Logic:** ML is the logic of the Kripke frames $(P(\{0, \ldots, n\}) - \{\{0, \cdots, n\}\}, \subseteq)$.
- ML was originally introduced as the characterization of Kolmogorov's logic of finite problems. Hence, it is expected to be a relevant logic here, as well.

The followings are the consequences of the previous theorem:

Corollary

- *Markov and Proof-irrelevant (Theory):* $\mathbf{T}_{MPI}(\text{IZF}) = \text{KP}^n = \text{KP} + \{\neg\neg p \to p \mid p \text{ is an atom}\}$.
- *Markov and Proof-irrelevant (Logic):* $\mathbf{L}_{MPI}(\text{IZF}) = \text{ML}$.
- *Kolmogorov:* $\mathbf{L}_K(\text{IZF}) = \text{ML}$.

BHK Interpretations and their Theories and Logics

Brouwer's Interpretation	**Theory**	**Logic**
without conditions	IPC	IPC
Proof-irrelevant	IPC	IPC
Markov (up to proof-irrelevancy)	IPC^n	IPC
Kolmogorov	?	IPC

Heyting's Interpretation	**Theory**	**Logic**
without conditions	above KP?	above KP?
Proof-irrelevant	INP	above KP?
Markov (up to proof-irrelevancy)	KP^n	ML
Kolmogorov	?	ML

Thank you for your attention!